Quasi-Schur Transformation for the Stable Compact Modeling of Piezoelectric Energy Harvester Devices

Siyang Hu, Chengdong Yuan, and Tamara Bechtold

Abstract The 'Schur after MOR' method has proved successful in obtaining stable reduced piezoelectric device models. Even though the method is already used in industry, the stability preservation of 'Schur after MOR' is still mathematically unproven. In this work, we show that the involved quasi-Schur transformation indeed does efficiently re-stabilize the aforementioned reduced piezoelectric energy harvester models. The transformation is only quasi-Schur as the unstable reduced systems require eigenspace projection and approximation to become Schur-transformable. During the transformation, the negative eigenvalues are eliminated from the reduced stiffness matrix and the system is stabilized. The mathematical proof is validated by numerical experiments on two different piezoelectric energy harvester devices. Furthermore, we compare 'Schur after MOR' to another recently presented stabilization method: 'MOR after Implicit Schur'. We show that the computational effort is significantly reduced.

1 Introduction

Modeling and simulation-driven development has become state-of-the-art due to the increasing capacity of today's computers. However, even the power of modern computers fails to always cope with the faster growing demands of the industry. To overcome this issue, the methodology of model order reduction (MOR) has been introduced. MOR significantly reduces the computational effort required for e.g. system-level simulations by replacing the original high-dimensional model with a lower dimensional but still accurate surrogate. Novel MOR methods are mostly

Siyang Hu[1,2], Chengdong Yuan[1,2], Tamara Bechtold[1,2]
[1]Jade University of Applied Sciences, Department of Engineering, Friedrich-Paffrath-Str.101, Wilhelmshaven 26389, Germany. [2]University of Rostock, Institute for Electronic Appliances and Circuits, Albert-Einstein-Str.2, Rostock 18059, Germany. e-mail: siyang.hu@jade-hs.de, chengdong.yuan@jade-hs.de, tamara.bechtold@jade-hs.de.

interpolation-based and perform well when applied to single-physical-domain models [3–5]. However, for models involving coupled physical domains, we encounter difficulties in preserving the stable input/output behavior of the original system [1].

In [1], the authors introduce three different approaches to solve stability issues they have encountered when reducing piezoelectric models. However, except for 'MOR after Schur' in [2], none of those methods have been mathematically proven yet. In this contribution, we considered the 'Schur after MOR' approach, as it proved effective in a number of industrial applications. We prove that the stable input/output behavior of the original system can be re-established by the quasi-Schur transformation involved in 'Schur after MOR'. The transformation is only quasi-Schur as an approximative pre-processing of the reduced model is required to make it Schur-transformable.

Section 2 briefly introduces a generic model of a piezoelectric energy harvester device. On this model, we recapture the 'Schur after MOR' approach in Section 3 while introducing some preliminaries on the way. Subsequently, we establish a link between the quasi-Schur transformation performed during 'Schur after MOR' and the stabilization of the reduced model. In Section 4, we present results of some numerical experiments. We validate our proof on two different harvester devices and show that 'Schur after MOR' is significantly more efficient than 'MOR after Schur' and its improved successor 'MOR after Implicit Schur', introduced in [2]. Finally, we conclude and give a brief outlook in Section 5.

2 Coupled Domain Finite Element Models of Piezoelectric Energy Harvesters

Piezoelectric energy harvesters transform environmental mechanical vibration into electrical energy using the piezoelectric effect [1, 6]. The mechanical part of the coupled domain finite element model reads:

$$\mathbf{M_{11}}\ddot{\mathbf{x}}_\mathbf{1} + \mathbf{D_{11}}\dot{\mathbf{x}}_\mathbf{1} + \mathbf{K_{11}}\mathbf{x_1} = \mathbf{b_1}u, \tag{1}$$

where $\mathbf{M_{11}}, \mathbf{K_{11}} \in \mathbb{R}^{n\times n}$ are the symmetric positive definite (s.p.d.) mass and stiffness matrices, respectively. $\mathbf{D_{11}} = \alpha\mathbf{M_{11}} + \beta\mathbf{K_{11}}, \alpha, \beta \in \mathbb{R}$ is the damping matrix and $\mathbf{x_1}$ is the vector of nodal displacements. The electrical part of the coupled domain finite element model reads:

$$\mathbf{K_{22}}\mathbf{x_2} = \mathbf{b_2}u, \tag{2}$$

with $\mathbf{K_{22}} \in \mathbb{R}^{k\times k}$ the electrical conductivity matrix, which is symmetric negative definite (s.n.d.) and

$$||\lambda_{\max}(\mathbf{K_{22}})|| \ll ||\lambda_{\min}(\mathbf{K_{11}})|| \tag{3}$$

holds for the respective eigenvalues. $\mathbf{x_2}$ is a vector of nodal electrical potentials. Both physical domains are coupled via piezoelectric patches, which transform vibrational stress into electric field. Thus, we have the piezoelectric coupling term $\mathbf{K_{12}} \in \mathbb{R}^{n\times k}$, such that:

$$\Sigma = \begin{cases} \underbrace{\begin{bmatrix} \mathbf{M_{11}} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{D_{11}} & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{K_{11}} & \mathbf{K_{12}} \\ \mathbf{K_{12}^T} & \mathbf{K_{22}} \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}}_{\mathbf{b}} u \\ y = \mathbf{c^T} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} \end{cases} . \quad (4)$$

The input function u corresponds to the displacement imposed to the harvester structure with input distribution vector $\mathbf{b} \in \mathbb{R}^{n+k}$ chosen accordingly. The total electrical potential is gathered via the output vector $\mathbf{c} \in \mathbb{R}^{n+k}$ within the output y.

3 Schur after MOR

This section briefly reassembles the 'Schur after MOR' procedure introduced in [1]. For a survey on general MOR methods for this class of models, please refer to [7].

'Schur after MOR' method stabilizes unstable reduced order models:

$$\Sigma_r = \begin{cases} \mathbf{M_r}\ddot{\mathbf{x}}_\mathbf{r} + \mathbf{D_r}\dot{\mathbf{x}}_\mathbf{r} + \mathbf{K_r}\mathbf{x_r} = \mathbf{b_r}u \\ y = \mathbf{c_r^T}\mathbf{x_r} \end{cases} , \quad (5)$$

obtained by projective MOR: $\mathbf{V^T}\Sigma\mathbf{V}$, where:

$$\begin{aligned} \{\mathbf{M_r}, \mathbf{D_r}, \mathbf{K_r}\} &= \mathbf{V^T}\{\mathbf{M}, \mathbf{D}, \mathbf{K}\}\mathbf{V}, \\ \mathbf{b_r} = \mathbf{V^T}\mathbf{b} \quad &\text{and} \quad \mathbf{c_r^T} = \mathbf{c^T}\mathbf{V}. \end{aligned} \quad (6)$$

$\mathbf{V} \in \mathbb{R}^{(n+k)\times p}, p \ll n+k$, is chosen as an orthonormal basis of the p-dimensional second-order input Krylov subspace via the second order Arnoldi reduction (SOAR) method [8,9]:

$$\mathscr{K}_p(-\mathbf{K^{-1}M}, -\mathbf{K^{-1}D}, -\mathbf{K^{-1}b}). \quad (7)$$

The stabilization of the reduced model is achieved by performing a quasi-Schur transformation on Σ_r, where Σ_r is approximated by a system of differential algebraic equations (DAEs) before being Schur transformed. The approximation involves an eigen-transformation $\widetilde{\Sigma}_r = \mathbf{T^T}\Sigma_r\mathbf{T}$, where $\mathbf{T}$ is a sorted orthonormal eigenbasis of the matrix $\mathbf{M_r}$, such that for the entries of $\widetilde{\mathbf{M}}_\mathbf{r}$, $\widetilde{m}_{r,ii} \geq \widetilde{m}_{r,jj}$ holds for all $i > j$. In the next step, we set $\widetilde{m}_{r,ii} = 0$ for all $i \geq I$ with $I \in [1,p]$, such that $\widetilde{m}_{r,(I-1)(I-1)} \gg \widetilde{m}_{r,II}$. In this way, we obtain a reduced order DAE system, which can be Schur transformed. We call the subspace spanned by all those eigenvectors corresponding to these eigenvalues $\widetilde{m}_{r,ii}, i \geq I$ quasi-algebraic.

In [8], a criteria for the stability of a second-order DAE is given.

Lemma 1 (Stability Criteria for Second-Order DAEs [8]). *A second-order DAE is stable, if* $D + D^T \succeq 0, M = M^T \succeq 0$ *and* $K = K^T \succ 0$.

Proof. The proof is given in [8].

With this criteria, we can prove that the quasi-Schur transformation stabilizes the reduced order system.

Theorem 1. *The quasi-Schur transformation stabilizes the reduced model* Σ_r.

Proof. As $\mathbf{M_{11}}$ is s.p.d., $\mathbf{M_r}$ has to be symmetric positive semi-definite as well. Furthermore, $\mathbf{K_r}$ must have negative eigenvalues. Otherwise, Σ_r is stable according to Lemma 1.

Since $\mathbf{M}$ and $\mathbf{K}$ can obviously be simultaneously diagonalized (e.g. with eigenbasis of $\mathbf{K^{-1}M}$), the system domain can be represented as a direct sum of these eigenspaces. Thus:

$$\lambda(\mathbf{K_r}) = \sum_i \nu_i \lambda(\mathbf{K})_i, \quad \sum_i \nu_i = 1, \tag{8}$$

holds for all eigenvalues of $\mathbf{K_r}$. Now, given (3) and let $P, N \subset \{1, ..., n+k\}$ be the set of indices corresponding to the positive and negative eigenvalues of $\mathbf{K}$. (8) can only be negative if $\sum_{i\in P} \nu_i \ll \sum_{i\in N} \nu_i$. That is to say, given the structure of Σ, the subspaces of the reduced system corresponding to these negative eigenvalues has to be dominated by the electric domain. Since $\mathbf{M}$ and $\mathbf{K}$ share the same decomposition, $\lambda(\mathbf{M_r}) = \sum_{i\in P} \nu_i \lambda(\mathbf{M})_i \approx 0$ must also hold.

Finally, when $\widetilde{\Sigma}_r$ is Schur transformed, we have:

$$\widetilde{\mathbf{K}}_\mathbf{s} = \widetilde{\mathbf{K}}_{\mathbf{r},(\mathbf{1:I},\mathbf{1:I})} - \widetilde{\mathbf{K}}_{\mathbf{r},(\mathbf{1:I},\mathbf{I:p})} \widetilde{\mathbf{K}}^{-\mathbf{1}}_{\mathbf{r},(\mathbf{I:p},\mathbf{I:p})} \widetilde{\mathbf{K}}_{\mathbf{r},(\mathbf{I:p},\mathbf{1:I})} \tag{9}$$

which is s.p.d as $\widetilde{\mathbf{K}}_{\mathbf{r},(\mathbf{1:I},\mathbf{1:I})}$ is s.p.d. and $\widetilde{\mathbf{K}}^{-\mathbf{1}}_{\mathbf{r},(\mathbf{I:p},\mathbf{I:p})}$ s.n.d.[1] This makes the quasi-Schur Transformed system stable according to Lemma 1.

Remark 1. In industrial software, the quasi-Schur Transformation introduced in [1] is actually modified [10]. The index I is obtained by the eigen-transformation $\widehat{\mathbf{K}}_\mathbf{r}$=$\mathbf{T_K^T K_r T_K}$ and then setting I such that $\widehat{K}_{r,ii} < 0$ for all $i \geq I$. This equivalent criteria is easier to implement and more robust.

[1] $\widetilde{\mathbf{K}}_{\mathbf{r},(\mathbf{1:I},\mathbf{1:I})}$ is the submatrix consisting of the first I rows and columns of $\widetilde{\mathbf{K}}_\mathbf{r}$, and $\widetilde{\mathbf{K}}_{\mathbf{r},(\mathbf{I:p},\mathbf{I:p})}$ consists of the the rows and columns I to p of $\widetilde{\mathbf{K}}_\mathbf{r}$.

4 Numerical Experiments

For the validation of Theorem. 1, the micro-structured energy harvester device from [1] (see Fig. 1 left) as well as a novel frequency tunable piezoelectric energy harvester introduced in [2] (see Fig. 1 right) are considered. Both energy harvester models are excited by a displacement input ***displ***. Beam sections carry piezoelectric patches that convert mechanical to electrical energy. Both models provide two electric ports named ***el1*** and ***el2***, which can be interfaced to electrical circuitries. To test the stability of the reduced model, the displacements at further nodes (***centre***, ***north***, ***south***) are selected as outputs of the system.

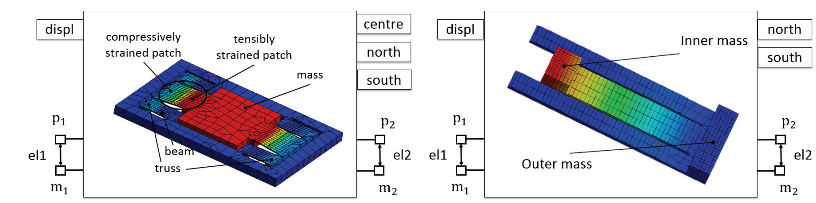

Fig. 1 micro-structured piezoelectric energy harvester model adopted from [1] (left); tunable piezoelectric energy harvester model adopted from [2] (right).

According to the observation from [1, 2], the reduced piezoelectric energy harvester model computed by conventional SOAR method shows unstable behavior at the electrical ports ***el1*** and ***el2***. Therefore, quasi-Schur transformation has been performed subsequently on these two reduced models to stabilize them. The accuracy of the respective frequency responses are shown in Fig. 2 and Fig. 3. The plots also include the reduced models obtained from 'MOR after Implicit Schur' from [2] for comparison.

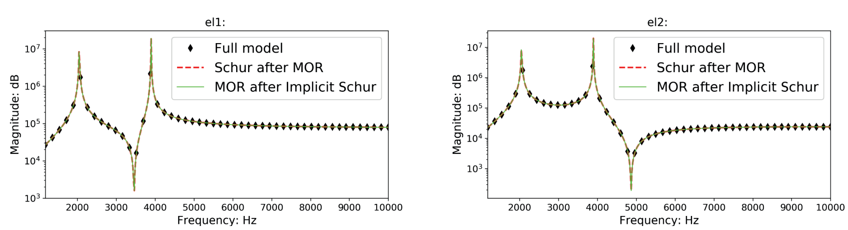

Fig. 2 Frequency response of the output voltage from ports ***el1*** and ***el2*** with displacement excitation ***displ*** of full and reduced micro-structured model from [1].

To validate Theorem 1, Algorithm. 1 is implemented. Various reduced models with different dimensions (6, 9, 30, 60, 120 and 240) have been tested. We found

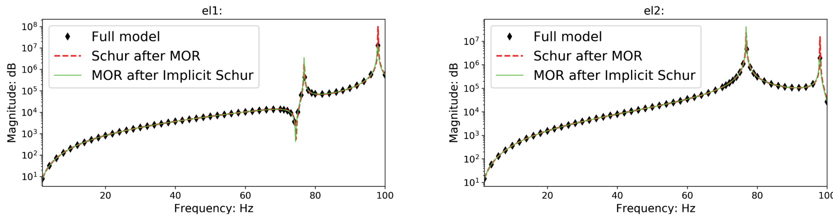

Fig. 3 Frequency response of the output voltage from ports ***el1*** and ***el2*** with displacement excitation ***displ*** of full and reduced tunable frequency harvester model from [2].

the considered subspaces to coincide (see Fig. 4, $\theta = 0°$), even when taking numeric errors into account. This validates Theorem 1 and justifies the assumption of a quasi-algebraic subspace.

Algorithm 1

Read reduced system matrices.
Check stability:
if $\{\mathbf{M_r}, \mathbf{D_r}, \mathbf{K_r}\} = \{\mathbf{M_r}, \mathbf{D_r}, \mathbf{K_r}\}^T$ and $\mathbf{M_r} \succeq 0, \mathbf{K_r} \succ 0$ and $\mathbf{D_r} + \mathbf{D_r}^T \succeq 0$ **then**
  reduced piezoelectric model is stable
  break
else
  reduced piezoelectric model is unstable
Negative eigenvector subspaces in $\mathbf{T_M}, \mathbf{T_K}$:
$\mathbf{T_M} \leftarrow$ eigenvector matrix of matrix $\mathbf{M_r}$
$\mathbf{T_K} \leftarrow$ eigenvector matrix of matrix $\mathbf{K_r}$
$I \leftarrow$ set the indices of negative eigenvalues in matrix $\widetilde{\mathbf{K_r}}$.
$\mathbf{T_{M,I}} \leftarrow$ columns of $\mathbf{T_M}$ indexed by I.
$\mathbf{T_{K,I}} \leftarrow$ columns of $\mathbf{T_K}$ indexed by I.
if $subspace\_angle(\mathbf{T_{M,I}}, \mathbf{T_{K,I}}) = 0$ **then**
  considered quasi-algebraic subspaces coincide.

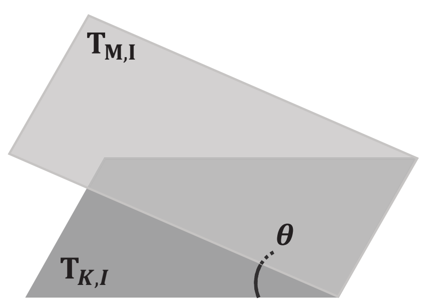

Fig. 4 Angle θ between the quasi-algebraic subspaces $\mathbf{T_{M,I}}$ and $\mathbf{T_{K,I}}$.

Table 1 shows the computation times of 'Schur after MOR' compared to 'MOR after Implicit Schur' from [2] for the reduction of the tunable piezoelectric energy harvester model with 24643 degrees of freedom. 'Schur after MOR' speeds up the computation of the reduced order model, as it avoids the implicit Schur transformation on the full model.

Table 1 Computation time of 'Schur after MOR' vs. 'MOR after implicit Schur'. (On Intel® Core™ i5-7600 CPU@3.5GHz, 32GB RAM)

Reduced Order	Schur after MOR	MOR after implicit Schur
30	36.98s	52.85s
90	41.48s	57.24s
240	57.42s	71.11s

5 Conclusion and Outlook

In this work, we have given a mathematical proof for stability preservation of 'Schur after MOR' method, which was initially suggested in [1]. We have shown that the quasi-Schur transformation, when applied to reduced models of piezoelectric energy harvesters obtained by projective MOR, stabilizes the models.

We have also shown its efficiency ($\sim$30% decrease of computation time) compared to 'MOR after Implicit Schur', which was initially suggested in [2].

In the next step, one can compare quasi-Schur transformation with conventional stabilization method, e.g. by simply truncating the unstable part of the reduced system. Finally, on can also consider comparing the performance of the whole 'Schur after MOR' procedure to structure preserving MOR, which is a third suggested stability preserving MOR method in [1].

References

1. Kudryavtsev, M., Rudnyi, E.B., Korvink, J.G., Hohlfeld, D., Bechtold, T.: Computationally efficient and stable order reduction methods for a large-scale model of mems piezoelectric energy harvester. Microelectronics Reliability, **55(5)**, 747–757 (2015)
2. Hu, S.Y., Yuan, C.D., Castagnotto, A., Lohmann, B., Bouhedma, S., Hohlfeld, D., Bechtold, T.: Stable reduced order modeling of piezoelectric energy harvesting modules using implicit schur complement. Microelectronics Reliability, **85**, 148–155 (2018)
3. Freund, R.W.: Krylov-subspace methods for reduced-order modeling in circuit simulation. J. Comput. Appl. Math. **123**, 395–421 (2000)
4. Gugercin, S., Stykel, T., Wyatt, S.: Model reduction of descriptor systems by interpolary projection methods SIAM J. Sci. Comput. **35(5)**, B1010–B1033 (2013)
5. Beattie, C., Gugercin, S.: Model Reduction by Rational Interpolation. Ch.7, pp. 297–334 (2017)

6. Bouhedma, S., Zheng, T.H., Hohlfled, D.: Multiphysics Modeling and Simulation of a Dual Frequency Energy Harvester. In: ECMS 2018 Proceedings 2018, pp. 386–390 (2018)
7. Benner, P., Stykel, T.: Model order reduction for differential-algebraic equations: a survey. In: Surveys in Differential-Algebraic Equations IV, pp. 107–160, Springer (2017)
8. Salimbahrami, S.B.: Structure preserving order reduction of large scale second order models. Ph.D. thesis, Technische Universitt Mnchen (2005).
9. Bai, Z., Su, Y.: Dimension reduction of large-scale second-order dynamical systems via a second-order Arnoldi method. SIAM J. Sci. Comput. **26(5)**, 1692–1709 (2005)
10. Rudnyi, E.B.: MOR for ANSYS. In: system-Level Modeling of MEMS, pp. 425–438, (2013)